# Explicit auto-Backlund relation through gauge transformation



1987 J. Phys. A: Math. Gen. 20 1107

(http://iopscience.iop.org/0305-4470/20/5/022)

J. Phys. A: Math. Gen. **20** (1987) 1107–1114. Printed in the UK

# Explicit auto-Bäcklund relation through gauge transformation

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Received 24 April 1986, in final form 30 July 1986

**Abstract.** Application of gauge transformation for generating explicit auto-Bäcklund relations is demonstrated for a number of integrable systems, e.g. KdV, SG, NLS, DNLS, mixed DNLS, modified DNLS, LLE, etc. This method turns out to be much simpler, effective and straightforward. Using Bäcklund relations all conserved quantities are expressed in a novel derivative-free form, which simplifies the calculation of important soliton characteristics.

## 1. Introduction

The importance of gauge transformation (GT) for connecting various non-linear systems is stressed in a number of publications [1-4]. In a series of papers [5-7] we also demonstrated the role of GT in generating new integrable systems and establishing the Bäcklund transformation (BT) between solutions of different non-linear equations. In this paper we aim to complement our previous work by extending the applicability of GT for deriving explicit auto-BT, i.e. for finding the relation between different solutions of the same equation. Though the general formulation of this idea is already available [8-10], application of this elegant approach to various systems of physical interest has not been demonstrated. We apply this method to find the explicit auto-Bäcklund relation (ABR) for different non-linear systems such as KdV, sine-Gordon (SG), the non-linear Schrödinger equation (NLS), derivative NLS (DNLS), the Landau–Lifshitz equation (LLE), etc. The ABR for a modified DNLS, which Boiti *et al* [11] had failed to discover, is found by us through some extension of this method. The ABR of mixed DNLS is also obtained similarly in a simple way. Using the BT, the order of derivatives connected with soliton solutions is lowered, which helps to rewrite all the conservation laws of integrable systems in a novel manner. These expressions become rather fascinating for a one-soliton solution and easily yield soliton momentum, energy and other integral characteristics in explicit form.

## 2. Examples of the explicit auto-Bäcklund relation through GT

Let the linear system corresponding to a non-linear equation $\hat{L}q = 0$ with solution $q$ take the form

$$\Psi_{x_i}(x_i, \lambda) = U_i(\lambda, q)\Psi(x_i, \lambda) \qquad i = 0, 1 \tag{2.1}$$

then the respective linear system for a different solution may be given by a gauge transformation of (2.1) as

$$\Psi'(x_i, \lambda) = G(q, q', \lambda)\Psi(x_i, \lambda) \tag{2.2}$$

 

which connects separate solutions $q$, $q'$ of the same non-linear equation

$$\{U_{1x_0} - U_{0x_1} + [U_1, U_0]\}|_{q=q'} = G\{U_{1x_0} - U_{0x_1} + [U_1, U_0]\}G^{-1} = 0 \tag{2.3}$$

in the implicit relation

$$G_{x_i}(q, q', \lambda) = U_i(q', \lambda)G(q, q', \lambda) - G(q, q', \lambda)U_i(q, \lambda). \tag{2.4}$$

For finding the explicit Bäcklund transformation for concrete examples, suppose

$$G = \begin{pmatrix} \alpha, \beta \\ \gamma, \delta \end{pmatrix} \qquad U_i = \begin{pmatrix} A_i, B_i \\ C_i, -A_i \end{pmatrix} \tag{2.5}$$

and rewrite (2.4) for the matrix elements:

$$\begin{aligned}
\alpha_{x_i} &= \alpha(A_i' - A_i) + \gamma B_i' - \beta C_i \\
\beta_{x_i} &= \beta(A_i' + A_i) + \delta B_i' - \alpha B_i \\
\gamma_{x_i} &= -\gamma(A_i' + A_i) + \alpha C_i' - \delta C_i \\
\delta_{x_i} &= -\delta(A_i' - A_i) + \beta C_i' - \alpha B_i
\end{aligned} \tag{2.6}$$

where $A_i' = A_i|_{q=q'}$, $B_i' = B|_{q=q'}$, etc, and $\alpha$, $\beta$, $\gamma$, $\delta$ are in general functions of $q$, $q'$ and $\lambda$. We will be interested in auto-BT between $N$ and $N+1$ solitons in different integrable systems. The $N$-soliton corresponds to the appearance of the $N$-pole in the penetration coefficient of the scattering matrix. Hence, to get the $(N+1)$-soliton solution from the $N$-soliton solution one has to 'add' another pole, which is achieved by the GT (2.1) through a spectral-parameter-dependent gauge matrix $G$. Dependence of $G$ on $\lambda$ is determined easily by the particular structure of the spectral matrix $U_1$. In particular, we may express $\alpha = \Sigma_{n=0}^{N}(\alpha_n/(\mathrm{i}\lambda)^n)$, $\beta = \Sigma_{n=0}^{N}(\beta_n/(\mathrm{i}\lambda)^n)$, etc, and from (2.6) choose $N$ by matching coefficients of equal powers in $(\mathrm{i}\lambda)^n$ to get a consistent closed set of equations for $\alpha_n$, $\beta_n$, $\gamma_n$ and $\delta_n$. The value of $N$ and explicit solutions would depend on a particular form of $U_1$.

### *2.1.* BT *for* AKNS *system*

For AKNS [12] spectral problem we have

$$A_1 = \mathrm{i}\lambda \qquad B_1 = q \qquad C_1 = r \tag{2.7}$$

which specifies $N = 1$ and yields $\beta_1 = \gamma_1 = \alpha_{1x} = \delta_{1x} = 0$, leading to the simple ansatz

$$\alpha = \alpha_0 + 2\mathrm{i}\lambda \qquad \beta = \beta_0 \qquad \gamma = \gamma_0 \qquad \delta = \delta_0 - 2\mathrm{i}\lambda \tag{2.8}$$

where $\alpha_0$, $\beta_0$, $\gamma_0$, $\delta_0$ are independent of $\lambda$. Now, putting (2.7) and (2.8) in relation (2.6) for $i = 1$ and comparing coefficients of equal powers of $\lambda$ we obtain

$$\begin{aligned}
&\beta_0 = q + q' \qquad \beta_{0x} = q'\delta_0 - q\alpha_0 \qquad \gamma_0 = r + r' \\
&\alpha_{0x} = (q'r' - qr) \qquad \delta_0 = \alpha_0 + e
\end{aligned} \tag{2.9}$$

with $e$ as a constant of integration, leading to the relation

$$\alpha_0 v = u_x - \tfrac{1}{2}e(u+v) \tag{2.10a}$$

$$\alpha_{0x} = \tfrac{1}{2}[r'(u+v) - r(u-v)] \tag{2.10b}$$

where $u \equiv q' + q$ and $v \equiv q' - q$ (2.10*a*, *b*) represent the space part of BT for the AKNS system, which reduces to an explicit form for different integrable equations with different choices of $q$ and $r$.

*2.1.1.* BT *for* KdV *equation.* The choice of $r=-1$, $q=-w_x$ in (2.7) gives the KdV equation

$$w_t+w_{xxx}-3w_x^2=0.$$

Introducing $\tilde{u}=w'+w$, $\tilde{v}=w'-w$ and assuming $\alpha_0=\alpha_0(\tilde{v})$ one gets from (2.10*b*) $\alpha_{0x}=\alpha_0'\tilde{v}_x=\tilde{v}_x$ or $\alpha_0=\tilde{v}$ and consequently, from (2.10*a*),

$$\tilde{u}_x=\tfrac{1}{2}\tilde{v}^2+\tfrac{1}{2}ew'-2\eta^2 \tag{2.11}$$

where $\eta$ is a constant of integration. Equation (2.11) yields for the choice of $e=0$ the conventional form of ABT

$$w_x+w_x'=\tfrac{1}{2}(w'-w)^2-2\eta^2. \tag{2.12}$$

To find the time ABT, insert $G$ with (2.8) in (2.6) with $i=0$. Using $U_0$ for KdV [13] one gets the equation

$$\alpha_{0t}=-\alpha_0(q_x'-q_x)-\gamma_0(q_{xx}'+2q'^2)-2q\beta_0$$

which, with the use of the obtained solutions,

$$\alpha_0=\delta_0=(w'-w) \qquad \beta_0=-(w_x'+w_x) \qquad \gamma_0=-2 \tag{2.13}$$

reduces immediately to the time BT

$$w_t'-w_t=-4\eta^2q-2q_x'(w'-w)+q'(w'-w)^2+2(q_{xx}'+2q'^2). \tag{2.14}$$

*2.1.2.* BT *for* SG *equation.* For $r=-q=\varphi_x$ the AKNS system (2.7) reduces to the SG equation $2\varphi_{xt}=\sin 2\varphi$. Assuming $\alpha_0=\alpha_0(\tilde{v})$ and $e=0$ as before, (2.10*a*, *b*) gives $\alpha_0'=-\tilde{u}_x$ and $\tilde{v}_x\alpha_0=\tilde{u}_{xx}$, respectively, which leads for $\alpha_0=f'$ to

$$f=\tilde{u}_x=-f'' \tag{2.15}$$

or $f''+f=0$, yielding a solution $f=2\eta\sin\tilde{v}$ or $\alpha_0=2\eta\cos\tilde{v}$. This solution gives directly from (2.15) the required space BT:

$$\varphi_x'+\varphi_x=2\eta\sin(\varphi'-\varphi). \tag{2.16}$$

With the obtained solution for $G$:

$$\alpha_0=2\eta\cos(\varphi'-\varphi)=\delta_0 \qquad \beta_0=-(\varphi_x'+\varphi_x)=-\gamma_0$$

and known $U_0$ for SG [13], it is now easy to extract the time BT from the equation $\alpha_{0t}=\frac{1}{2}(\cos 2\varphi-\cos 2\varphi')$ in the form

$$\varphi_t'-\varphi_t=(1/2\eta)\sin(\varphi'+\varphi). \tag{2.17}$$

*2.1.3.* BT *for* NLS *equation.* The reduction of AKNS with $r=\varepsilon q^*$, $\varepsilon=\mp 1$ yields the NLS equation $\mathrm{i}q_t+q_{xx}-\varepsilon|q|^2q=0$, which leads (2.10) to

$$a_x=\tfrac{1}{2}\varepsilon(uv^*+vu^*) \qquad \mu_x=0 \tag{2.18}$$

where $\alpha_0=a+2\mathrm{i}\mu$. Assuming $a=a(|u|^2)$ only and choosing the constant $c=-4\mathrm{i}\mu$ we obtain from (2.10*a*) and (2.18) $(a^2)'=\varepsilon$, yielding the solution $a=-(4\nu^2+\varepsilon|u|^2)^{1/2}$, where $4\nu^2$ is the integration constant. (2.10*a*) reduces now to $u_x=\alpha_0 v-2\mathrm{i}\mu(u+v)$, which leads to the space BT:

$$q_x'+q_x=-2\mathrm{i}\mu(q'+q)-(q'-q)(4\nu^2+\varepsilon|q'+q|^2)^{1/2}. \tag{2.19}$$

Therefore, the gauge $G$ is given by (2.8) with

$$\begin{aligned}\alpha_0&=2\mathrm{i}\mu-(4\nu^2+\varepsilon|u|^2)^{1/2}=\delta_0^*\\ \beta_0&=q'+q=-\gamma_0^*\end{aligned} \tag{2.20}$$

which gives with the known [13] time evolution operator $U_0$ for NLS the explicit time BT:

$$i(q'_t+q_t)=\varepsilon(q'+q)(|q'|^2+|q|^2)+(q'_x-q_x)(4\nu^2+\varepsilon|q'+q|^2)^{1/2}+2i\mu(q'_x+q_x). \tag{2.21}$$

### 2.2. BT for Kaup-Newell problem: DNLS equation

The Kaup and Newell [14] spectral problem is given by

$$A_1=-i\lambda^2 \qquad B_1=\lambda q \qquad C_1=\lambda r \tag{2.22}$$

which through (2.6) dictates $N=2$ and gives $\alpha_{0x}=\delta_{0x}=\alpha_1=\delta_1=\beta_0=\beta_2=\gamma_0=\gamma_2=0$ allowing the choice

$$\alpha=1+\alpha_2\lambda^2 \qquad \beta=\beta_1\lambda \qquad \gamma=\gamma_1\lambda \qquad \delta=-1+\delta_2\lambda^2 \tag{2.23}$$

$\alpha_2$, $\delta_2$, $\beta_1$, $\gamma_1$ being independent of $\lambda$. Repeating a similar procedure to the above, we get, from (2.23) and (2.6), the relation

$$\begin{aligned} &\beta_{1x}=-(q'+q) \qquad \beta_1=-\tfrac{1}{2}i(q'\delta_2-q\alpha_2)\\ &\alpha_{2x}=f\alpha_2 \qquad \delta_{2x}=-f\delta_2 \end{aligned} \tag{2.24}$$

where $f=\frac{1}{2}i(q'r'-qr)$. Equation (2.24) may be reduced further to

$$i(q'_x\delta_2-q_x\alpha_2)=-2(q'+q)+\tfrac{1}{2}if(q'\delta_2+q\alpha_2). \tag{2.25}$$

For the DNLS equation

$$iq_t+q_{xx}-i\varepsilon(|q|^2q)_x=0 \tag{2.26}$$

we need the reduction $r=\varepsilon q^*$, $\varepsilon=\pm1$, which yields from (2.25) the space part of BT for the DNLS:

$$i(q'_xa^*-q_xa)=-2(q'+q)-\tfrac{1}{2}\varepsilon(q'a^*+qa)(|q'|^2-|q|^2) \tag{2.27}$$

where $a=\rho\exp[i(\mu'^+-\mu^+)]$, $\mu^+=\frac{1}{2}\varepsilon\int_{-\infty}^x|q|^2\,dx$ and $\rho=-4\Delta^2e^{i\theta}$. One may also derive a one-soliton solution directly from (2.27) by setting $q=0$:

$$iq_x=2q\rho\,e^{-i\mu^+}+\tfrac{1}{2}\varepsilon|q|^2q$$

which after intergrating once gives the one-soliton solution

$$q=\pm4\Delta\sin\gamma\frac{\{\exp[(2\eta-2i\xi)x-i\mu^+]\}}{\exp4\eta x+\exp(\pm i\gamma)} \tag{2.28}$$

for $\varepsilon=-1$ and with $\eta=\Delta^2\sin\gamma$, $\xi=\mp\Delta^2\cos\gamma$. The time part of BT may also be found as before.

### 2.3. BT of the mixed DNLS equation

Mixed DNLS, which is a hybrid of the DNLS and NLS equation, given by [15],

$$iQ_t+Q_{xx}-\varepsilon\beta|Q|^2Q-i\varepsilon\alpha(|Q|^2Q)_x=0 \qquad \varepsilon=\mp1,\ \alpha>0,\ \beta>0 \tag{2.29}$$

corresponds to a choice $A_1=i[-\alpha\lambda^2+(2\beta)^{1/2}\lambda]$, $B_1=(\alpha\lambda-\sqrt{\beta/2})Q$ and $C_1=-\varepsilon B_1^*$. The above procedure may also be repeated here, but this leads to severe complications. However, it is interesting to notice that a simple gauge transformation

$$q_D=\alpha^{1/2}Q\exp[-\tfrac{1}{2}iv(x-\tfrac{1}{2}vt)] \tag{2.30}$$

with $v = 2\beta/\alpha$ and a Galilean transformation $x' = x - vt$, $t' = t$ maps the mixed DNLS solution $Q$ to the DNLS solution $q_D$. A direct substitution of (2.30) shows clearly that, if $Q$ is a solution of (2.29), $q_D$ must be a solution of (2.26). Therefore, ABT (2.27) of DNLS leads readily to a auto-Bäcklund transformation of mixed DNLS for the simple change of variables (2.30).

### 2.4. BT of WKI problem: modified DNLS equation

The Wadati-Konno-Ichikawa [16] spectral problem may be given by $U_1$ with

$$A_1 = -\mathrm{i}\lambda \qquad B_1 = \lambda\tilde{q} \qquad C_1 = \lambda\tilde{r} \tag{2.31}$$

which for the choice $\tilde{r} = -\tilde{q}^*$ yields the modified DNLS

$$\mathrm{i}\tilde{q}_t + \left(\frac{\tilde{q}}{(1+|\tilde{q}|^2)^{1/2}}\right)_{xx} = 0. \tag{2.32}$$

However, this slight modification of spectral structure forbids one to use directly the GT method discussed above and this is the cause of the failure of an earlier attempt [11] to find the BT for such systems. We, however, overcome this difficulty by extending the GT through the Landau-Lifshitz equation (LLE). For this we first found BT for the LLE

$$S_t = \frac{1}{2\mathrm{i}}[S, S_{xx}] \qquad S^2 = \mathbb{1} \qquad S = \boldsymbol{S}\cdot\boldsymbol{\sigma}. \tag{2.33}$$

It is known [1] that the LLE may be gauge generated from NLS through the transformation $\Phi = \Psi_0^{-1}\Psi$ where $\Phi$, $\Psi$ are Jost functions corresponding to the LLE and NLS, respectively, and $\Psi_0 = \Psi(\lambda = 0)$. Therefore, for a different solution $S'$ of LLE we have

$$\Phi' = \Psi_0'^{-1}\Psi' = \Psi_0^{-1}G_0^{-1}G\Psi = B\Phi$$

where $B = \Psi_0^{-1}G_0^{-1}G\Psi_0$ and $G_0 = G(\lambda = 0)$, i.e. $B$ is the BT gauge of the LLE, while $G$ is that of the NLS. Using the structure (2.8) for $G$ we obtain

$$B = \mathbb{1} + 2\mathrm{i}\lambda B_1 \qquad B_1 = \Psi_0^{-1}G_0^{-1}\sigma_3\Psi_0 \tag{2.34}$$

where $\Psi_0^{-1}\sigma_3\Psi_0 = S = \boldsymbol{S}\cdot\boldsymbol{\sigma}$ and $G_0$ is given by (2.20). From the relation (2.4) for gauge matrix $B$ with

$$U_1 = \mathrm{i}\lambda S \qquad U_0 = 2\mathrm{i}\lambda^2 S + 2\lambda SS_x$$

one obtains the ABT between different solutions $S'$ and $S$ of the LLE in the form

$$S' = B_1 S B_1^{-1} \tag{2.35$a$}$$

as the space part and

$$\mathrm{i}B_{1t} = (S'S'_x - SS_x) \tag{2.35$b$}$$

as the time part. Now, for finding the explicit relation, one has to express $B$ given by (2.34) as a function of $S$ and $S'$ by noting

$$\Psi_0 = \begin{pmatrix} (S^3-1)\rho & S^-\rho \\ -S^+\rho^* & (S^3-1)\rho^* \end{pmatrix} \qquad |\rho|^2 = \text{constant}(1-S^3)^{-1}, \arg\rho = \theta/2 \tag{2.36}$$

and

$$G_0=\begin{pmatrix}\alpha_0 & \beta_0\\ -\beta_0^* & \alpha_0^*\end{pmatrix}\qquad \alpha_0=2\mathrm{i}\mu-(4\nu^2-|q'+q|^2)^{1/2}\qquad \beta_0=q'+q \tag{2.37}$$

with the relation

$$q=|q|\,\mathrm{e}^{\mathrm{i}\theta}\qquad |q|=\tfrac{1}{2}(S_x\cdot S_x)^{1/2}\qquad \theta=\frac{\mathrm{i}}{2}\int_{-\infty}^{x}\frac{S_x^-S^+-S^-S_x^+}{(1-S^3)}\,\mathrm{d}x. \tag{2.38}$$

Here $S$ takes values in a manifold: $S\in \mathrm{SU}(2)$. We may, however, get another integrable equation

$$\mathrm{i}Q_t=(1-|Q|^2)^{1/2}Q_{xx}-Q(1-|Q|^2)^{1/2}_{xx} \tag{2.39}$$

from (2.33) defined in a vector space, by a change of variable

$$Q=S^-\qquad Q^*=S^+\qquad S^3=P\equiv(1-|Q|^2)^{1/2}. \tag{2.40}$$

The corresponding ABT may be obtained clearly from (2.35) by the substitution of (2.40).

We are now in a position to derive the ABT of the modified DNLS (2.32). It is remarkable that a change of dependent as well as independent variables [4] given by

$$\mathrm{i}\tilde{q}=Q/P\qquad X=-\int P\,\mathrm{d}x\qquad P=(1-|Q|^2)^{1/2} \tag{2.41}$$

transforms (2.39) to (2.32). Therefore, using

$$S=M(\tilde{q})=\tilde{p}\left(\sigma_3+\mathrm{i}\begin{pmatrix}0 & \tilde{q}\\ -\tilde{q}^* & 0\end{pmatrix}\right)\qquad \tilde{p}=(1+|\tilde{q}|^2)^{1/2}$$

one gets from (2.35) the Bäcklund transformation for (2.32) in the form

$$\tilde{p}'\begin{pmatrix}0 & \tilde{q}'\\ -\tilde{q}^{*\prime} & 0\end{pmatrix}=\tilde{p}\tilde{B}_1\begin{pmatrix}0 & \tilde{q}\\ -\tilde{q}^* & 0\end{pmatrix}\tilde{B}_1^{-1} \tag{2.42}$$

as the space part and

$$\mathrm{i}\tilde{B}_{1t}=(M'M'_x-MM_x) \tag{2.43}$$

as the time part, where $\tilde{B}_1=g^{-1}G_0^{-1}\sigma_3 g$ with

$$g=\begin{pmatrix}(\tilde{p}-1)\rho & \mathrm{i}\tilde{p}\tilde{q}\rho\\ \tilde{p}q^*\rho^* & (1-\tilde{p})\rho^*\end{pmatrix}$$

and $G_0$ is given by (2.37) with

$$|q|=\frac{\mathrm{i}\tilde{p}}{2(\tilde{p}-1)}(\tilde{p}^2\tilde{q}_x-(\tilde{p}\tilde{q})_x)\qquad \theta=-\frac{\mathrm{i}}{2}\int_{-\infty}^{X}\frac{\tilde{p}^2(\tilde{q}^*\tilde{q}_x-\tilde{q}\tilde{q}_x^*)}{1-\tilde{p}}\,\mathrm{d}x.$$

## 3. Expressions of conservation laws without derivative terms

It is well known [17] that the function $\Gamma=\Psi_2/\Psi_1$ expressed through the Jost function $\Psi=\binom{\Psi_1}{\Psi_2}$ reduces the spectral problem for the AKNS system, i.e. (2.1) with $i=1$, to a Ricatti equation

$$2\mathrm{i}\Gamma_{m+1}=(\Gamma_m)_x+\mathrm{i}r\sum_{i=1}^{m-1}\Gamma_{m-i}\Gamma_i\qquad \Gamma=\sum_{m=1}^{\infty}\Gamma_m\lambda^{-m} \tag{3.1}$$

with the solutions

$$\Gamma_1 = -\tfrac{1}{2}q \qquad \Gamma_2 = \tfrac{1}{4}\mathrm{i}q_x \qquad \Gamma_3 = \tfrac{1}{8}(q_{xx} + rq^2) \qquad \text{etc.} \tag{3.2}$$

The gauge transformation (2.2) changes $\Gamma$ to

$$\Gamma' = (\alpha\Gamma + \beta)/(\gamma\Gamma + \delta) \tag{3.3}$$

which gives the space BT between $\Gamma'_i$ and $\Gamma_i$ in the form

$$2\mathrm{i}(\Gamma'_{m+1} + \Gamma_{m+1}) - (\delta_0\Gamma'_m - \alpha_0\Gamma_m) - \gamma_0 \sum_{i=1}^{m-1} \Gamma'_i\Gamma_{m-i} = 0 \tag{3.4}$$

with $\Gamma'_1 + \Gamma_1 = \frac{1}{2}\mathrm{i}\beta_0$. Now, let $\Gamma'_i$ correspond to a $N$-soliton solution, then repeated use of (3.4) yields a reduction of higher-order terms through lower orders as

$$2\mathrm{i}\Gamma^{(N)}_{m+1} = \sum_{M=1}^{N} (-1)^{N-M}\left(\Gamma^{(M)}_m(\alpha_0 + \delta_0) + \gamma_0 \sum_{i=1}^{m-1} \Gamma^{(M)}_i \Gamma^{(M-1)}_{m-i}\right) - \alpha_0\Gamma^{(N)}_m. \tag{3.5}$$

$\Gamma^{(N)}_m$ with higher index $m$, as seen from (3.1), contains higher derivative terms. Therefore, repeated use of (3.5) would reduce the expression for conserved quantities (for the $N$-soliton)

$$I^{(N)}_m = \tfrac{1}{2}\mathrm{Re} \int_{-\infty}^{\infty} r^{(N)}\Gamma^{(N)}_m \,\mathrm{d}x \qquad m = 1, 2, 3, \ldots \tag{3.6}$$

to a derivative-free form, though containing contributions from all solitonic modes $M = 1, 2, \ldots, N$. For the one-soliton solution (since $\Gamma^{(0)}_m = 0$) this expression takes a simple interesting form

$$I^{(1)}_m = \tfrac{1}{2}\mathrm{Re} \int_{-\infty}^{\infty} r^{(1)} \left(\frac{\delta_0}{2\mathrm{i}}\right)^{m-1} \Gamma^{(1)}_1 \,\mathrm{d}x \tag{3.7}$$

where $\Gamma_1 = -\frac{1}{2}q$ for the AKNS systems. In the case of the NLS equation using (2.20), one gets a derivative-free expression for conservation laws of the one-soliton solution $q_s$ as

$$\begin{aligned} I_m &= \tfrac{1}{2}\mathrm{Re} \int_{-\infty}^{\infty} |q_s|^2 \left\{\mu + \frac{\mathrm{i}}{2}(4\nu^2 - |q_s|^2)^{1/2}\right\}^{m-1} \mathrm{d}x \\ &= \tfrac{1}{2}(-\mu)^{m-1} \int_{-\infty}^{\infty} |q_s|^2 \left(1 - \sum_{k=1}^{[(m-1)/2]} c^{m-1}_{2k} (2\mu)^{-2k} (4\nu^2 - |q_s|^2)^k\right) \mathrm{d}x \end{aligned} \tag{3.8}$$

which yields momentum and energy expressions with $m = 2$ and $m = 3$, respectively

$$P = c_2 = -\frac{\mu}{2} \int_{-\infty}^{\infty} |q_s|^2 \,\mathrm{d}x \tag{3.9a}$$

$$H = c_3 = \tfrac{1}{2} \int_{-\infty}^{\infty} |q_s|^2(\mu^2 - \nu^2 + |q_s|^2) \,\mathrm{d}x \tag{3.9b}$$

and reduces for the explicit soliton solution $|q_s|^2 = 4\nu^2 \operatorname{sech}^2 2\nu(x - vt)$ immediately to

$$P = -2\mu\nu = Nv \qquad H = \tfrac{2}{3}\nu(3\mu^2 - \nu^2) = -\tfrac{2}{3}N^3 + \tfrac{1}{2}Nv^2 \tag{3.10}$$

where $N = \nu$ is interpreted as the soliton 'mass' and $v = -2\mu$ as the soliton 'velocity'. This shows consequently $Nv$ as 'momentum', $\frac{1}{2}Nv^2$ as 'kinetic' and $-\frac{2}{3}N^3$ as 'rest' energies. Note that (3.10) reproduces in a simple way the known results [13] through action-angle variables. In a similar manner, we may deduce derivative-free expressions of conserved quantities for soliton solutions to the KdV, modified KdV, DNS, etc, equations. For example, in the case of KdV given by $r=-1$, $q=-w_x$, the soliton energy may be expressed as $H=\frac{1}{8}\int_{-\infty}^{\infty} w^2 q \, dx$.

## 4. Conclusion

Gauge transformation is applied to extract the explicit auto-Bäcklund relation connecting different solutions of some well known non-linear equations, e.g. KdV, SG, NLS, DNLS, LLE, etc. The use of GT for linking different systems helps to find ABT for mixed DNLS and also for a modified DNLS, where an earlier attempt failed. Though many of the relations found here have been obtained previously by various other methods [18-21], the present approach demonstrates elegantly how the GT works in a rather straightforward way and, compared to standard tricks for obtaining ABT, it is much simpler and free from any 'guesswork'. Along with the BT between solutions, we also find the BT between Jost functions, which helps in lowering the order of derivative terms connected with the $N$-soliton solution. This in turn rewrites expressions for infinite conserved quantities including momentum, energy, etc, of a soliton in a novel derivative-free form which, apart from its own appeal, also simplifies the explicit calculations. It is hoped that this GT method would also 'work' successfully in other systems not covered here.

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